The Jacobian Conjecture$_{2n}$ implies the Dixmier Problem$_n$

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Abstract

Using the *inversion formula* for automorphisms of the Weyl algebras with polynomial coefficients and the *bound* on its degree [1] a slightly shorter (*algebraic*) proof is given of the result of A. Belov-Kanel and M. Kontsevich [2] that JC_{2n} *implies* DP_n. No originality is claimed.

Mathematics subject classification 2000: 14R15, 14H37, 16S32.

The *Weyl* algebra $A_n = A_n(\mathbb{Z})$ is a $\mathbb{Z}$-algebra generated by $2n$ generators $x_1, \ldots, x_{2n}$ subject to the defining relations:

$$[x_{n+i}, x_j] = \delta_{ij}, \;\; [x_i, x_j] = [x_{n+i}, x_{n+j}] = 0 \;\text{ for all }\; i, j = 1, \ldots, n,$$

where δ_{ij} is the Kronecker delta, $[a, b] := ab - ba = (\mathrm{ad}\, a)(b)$. For a ring R, $A_n(R) := R \otimes_{\mathbb{Z}} A_n$ is the Weyl algebra over R.

- **The Jacobian Conjecture** (JC_n): given $\sigma \in \mathrm{End}_{\mathbb{C}-alg}(K[x_1, \ldots, x_n])$ such that $\det(\frac{\partial \sigma(x_i)}{\partial x_j}) \in K^* := K\backslash\{0\}$ then $\sigma \in \mathrm{Aut}_{\mathbb{C}}(K[x_1, \ldots, x_n])$.
- **The Dixmier Problem** (DP_n), [3]: is a $\mathbb{C}$-algebra endomorphism of the Weyl algebra $A_n(\mathbb{C})$ an algebra automorphism?

Theorem 1 *[1]* (The Inversion Formula) *For each $\sigma \in \mathrm{Aut}_K(A_n(K))$ and $a \in A_n(K)$,*

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^{2n}} \phi_\sigma(\frac{(\partial')^\alpha}{\alpha!} a) x^\alpha,$$

where $x^\alpha := (x_1')^{\alpha_1} \cdots (x_{2n}')^{\alpha_{2n}}$, $(\partial')^\alpha := (\partial_1')^{\alpha_1} \cdots (\partial_{2n}')^{\alpha_{2n}}$, $\partial_i' := \mathrm{ad}(\sigma(x_{n+i}))$ and $\partial_{n+i}' := -\mathrm{ad}(\sigma(x_i))$ for $i = 1, \ldots, n$, $\phi_\sigma := \phi_{2n}\phi_{2n-1} \cdots \phi_1$ where $\phi_i := \sum_{k \geq 0} (-1)^i \frac{(\sigma(x_i))^k}{k!} (\partial_i')^k$.

Remark. This result was proved when K is a field of characteristic zero, but by the *Lefschetz principle* it also holds for any commutative reduced $\mathbb{Q}$-algebra.

Theorem 2 *[1] Given $\sigma \in \mathrm{Aut}_K(A_n(K[x_{2n+1}, \ldots, x_{2n+m}]))$ where K is a commutative reduced $\mathbb{Q}$-algebra. Then the degree* $\deg \sigma^{-1} \leq (\deg \sigma)^{2n+m-1}$.

Theorem 3 *[2]* $JC_{2n} \Rightarrow DP_n$.

Proof. Let $\sigma \in \mathrm{End}_{\mathbb{C}-alg}(A_n(\mathbb{C}))$.

Step 1. Let R be a finitely generated (over $\mathbb{Z}$) $\mathbb{Z}$-subalgebra of $\mathbb{C}$ generated by the coefficients of the elements $x_i' := \sigma(x_i)$, $i = 1, \ldots, 2n$. Localizing at finitely many primes $q \in \mathbb{Z}$ one can assume that the ring $R_p := R/(p)$ is a domain for all primes $p \gg 0$. Then $\sigma \in \mathrm{End}_{R-alg}(A_n(R))$, $x_i' \in A_n(R) = R \otimes_{\mathbb{Z}} A_n$, and the centre $Z(A_n(R)) = R$.

Step 2. From this moment on $p \in \mathbb{Z}$ is any (all) sufficiently big prime number and $\mathbb{Z}_p := \mathbb{Z}/(p)$.

$$\begin{aligned} A(p) &:= A_n(R)/(p) \simeq R_p \otimes_{\mathbb{Z}_p} A_n(\mathbb{Z}_p) \simeq R_p \otimes_{\mathbb{Z}_p} M_{p^n}(\mathbb{Z}_p[x_1^p, \ldots, x_{2n}^p]) \\ &\simeq M_{p^n}(R_p[x_1^p, \ldots, x_{2n}^p]) = M_{p^n}(C_p) \end{aligned}$$

where x_i^p stands for $x_i^p + (p)$, and $M_{p^n}(C_p)$ is a matrix algebra (of size p^n) with coefficients from a polynomial algebra $C_p := R_p[x_1^p, \ldots, x_{2n}^p]$ over R_p. The σ induces an R_p-algebra endomorphism $\sigma_p : A(p) \to A(p)$, $a + (p) \mapsto \sigma(a) + (p)$.

Step 3. It follows from the inversion formula (Theorem 1) and Theorem 2 that

$$\sigma \in \mathrm{Aut}_R(A_n(R)) \quad \Leftrightarrow \quad \sigma_p \in \mathrm{Aut}_{R_p}(A(p)) \;\text{ for all }\; p \gg 0.$$

Step 4. $\sigma_p(C_p) \subseteq C_p$ (see [4]).

Step 5. Since $A(p) \simeq M_{p^n}(C_p)$, $Z(A(p)) = C_p$, and $\sigma_p(C_p) \subseteq C_p$, it is obvious that

$$\sigma_p \in \mathrm{Aut}_{R_p}(A(p)) \quad \Leftrightarrow \quad \sigma_p|_{C_p} \in \mathrm{Aut}_{R_p}(C_p).$$

Step 6. Claim: $\sigma_p(C_p) \subseteq C_p$ and JC_{2n} imply $\sigma_p|_{C_p} \in \mathrm{Aut}_{R_p}(C_p)$.

Proof of the Claim. (i). $(C_p, \{\cdot, \cdot\})$ is a *Poisson algebra* where

$$\{a + (p), b + (p)\} := \frac{[a, b]}{p} \pmod p$$

is the *canonical Poisson bracket* on a polynomial algebra in $2n$ variables (a direct computation, see Lemma 4, [2]) which is obviously σ_p-*invariant.*

(ii).

$$\begin{aligned} \{\pm 1\} &\ni \sigma_p(\det(\{x_i^p, x_j^p\})_{1 \le i,j \le n}) = \det(\sigma_p(\{x_i^p, x_j^p\})) \\ &= \det(\{\sigma_p(x_i^p), \sigma_p(x_j^p)\}) = \det(J^T(\{x_i^p, x_j^p\})J) \\ &= \det(J)^2 \det(\{x_i^p, x_j^p\})) = \det(J)^2 \cdot (\pm 1). \end{aligned}$$

where $J := (\frac{\partial \sigma(x_i^p)}{\partial (x_j^p)})_{1 \le i,j \le n}$. Hence, $\det(J) \in \{\pm 1\}$. Only *now* we use the assumption that JC_{2n} holds: which implies $\sigma_p|_{C_p} \in \mathrm{Aut}_{R_p}(C_p)$. $\square$

References

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